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Mathematics in the making: Mapping verbal discourse in Pólya's "Let Us Teach Guessing" lesson

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Abstract

This paper describes a detailed analysis of verbal discourse within an exemplary mathematics lesson—that is, George Pólya teaching in the Mathematics Association of America [MAA] video classic, "Let Us Teach Guessing" (1966). The results of the analysis reveal an inductive model of teaching that represents recursive cycles rather than linear steps. The lesson begins with a frame of reference and builds meaning cyclically/recursively through inductive processes—that is, moving from specific cases, through recursive cycles, toward more general hypotheses and rules. Additionally, connections to specific forms of talk and verbal assessment, as well as to univocal (conveying meaning) and dialogic (new meaning through dialogue) discourse, are made.

Keywords: Mathematical discourse; Problem solving

> "First, guess; then prove . . . Finished mathematics consists of proofs, but mathematics in the making consists of guesses" (Pólya, 1966).

1. Introduction

The National Council of Teachers of Mathematics [NCTM] has consistently recognized communication as an essential component of reform-oriented mathematics education (NCTM, 1989, 1991, 2000). However, talk alone is not sufficient; the *quality* and *type* of discourse affect its potential for promoting conceptual understanding (Kazemi & Stipek, 2001; Lampert & Blunk, 1998; Nathan & Knuth, 2003; van Oers, 2002; Van Zoest & Enyart, 1998). While the teacher's role has been found to be critical in how discourse plays out in a mathematics classroom (Chazan & Ball, 1995), current research demonstrates that even among teachers who report agreement with reform ideas, instructional practices emphasize routine procedures and repeated practice, giving learners little opportunity to investigate, conjecture, reason, and justify (Jacobs et al., 2006; National Center for Educational Statistics [NCES], 1999, 2000, 2001; Spillane & Zeuli, 1999; Stigler & Hiebert, 1997, 1998, 1999). Familiarity with reform ideas is not enough; indeed, there is evidence to suggest that the teaching professional lacks "knowledge about what constitutes effective teaching" (Stigler & Hiebert,

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1999, p. 12). Careful analysis of effective practices for orchestrating discourse in mathematics classrooms has the potential to provide specific knowledge that may enhance teaching and learning. With this end in mind, this paper describes a detailed analysis of discourse within an exemplary mathematics lesson—that is, George Pólya teaching in the Mathematics Association of America [MAA] video classic, "Let Us Teach Guessing" (1966). Additionally, the results of the analysis are connected to an associated inductive model of teaching, thus providing a focus on discourse as a means of promoting meaningful teaching and learning of mathematics.

2. Background

Decades before current mathematics reform documents were published, similar themes were espoused by George Pólya, noted mathematician, mathematics educator, and problem solving expert, in works such as "How to Solve It" (1945/1985), "Mathematical Discovery" (1962/1981) and his "Mathematics and Plausible Reasoning" volumes (1954a, 1954b). Pólya articulated a vision that emphasized active, engaging learning where the mathematics teacher's job is to help students "discover by themselves as much as feasible" (1962/1981, p. 104). He advocated that mathematics teachers use challenging, nonroutine problems as vehicles for developing understanding of mathematical concepts and, further, that they be facilitated through phases of exploration, formalization, and assimilation. Pólya noted that mathematics instruction typically involves "almost exclusively routine examples" (p. 106) that illustrate and offer practice, but that address only isolated rules—missing both the exploration and assimilation phases of learning. Selecting a rich problem is not enough, however; the teacher needs skills, strategies, and dispositions to help students not only to find solutions to problems, but also to recognize general patterns and to develop problem solving "know how" (1962/1981). Pólya's writings are instructive; seeing him "in action" further illustrates his ideas. This research unpacks the verbal discourse within an example of Pólya's *actual teaching practices*; thus, serving as a means of understanding teaching expertise and, in turn, effective teaching practices.

In 1965, Pólya was filmed teaching a mathematics lesson to a group of university students (1966). He began the lesson with a rich problem that was unfamiliar to the students—that is, *into how many parts is space divided by 5 planes*? During the lesson, Pólya used discourse to guide the students through cycles of evidence-based guesses (i.e., plausible reasoning), investigations, and explanations that resulted in mathematical sense-making—about the problem itself, about generalizations of the problem, and about strategies for approaching problem-solving. Pólya's teaching, and this lesson in particular, have stood the test of time as exemplars. In fact, other researchers have investigated the lesson; in particular, Leinhardt and Schwarz noted, "The advantage of examining this particular episode of teaching is that the mathematical and epistemological knowledge carried by the teacher in this lesson can justifiably be considered exemplary" (Leinhardt & Schwarz, 1997, p. 397). Leinhardt and Schwarz focused on "the instructional explanation of guessing as a heuristic for solving the Five Planes Problem" (1997, p. 305). This paper extends this research by examining the *verbal discourse* and its implications for teaching and learning associated with "mathematics in the making"[1] (Speiser & Walter, 2000).

While this research draws from divergent theoretical viewpoints, sociocultural theory, with its contention that higher mental functions derive from social interaction, provides a meaningful framework for analysis and discussion of discourse as a mediating tool in the teaching-learning process (Vygotsky, 1978, 1986/2002). When considering language as a mediator of meaning, it is useful to take into account the two main functions of communication—that is, univocal (one-way communication used to convey meaning) and dialogic (give-and-take communication used to generate new meaning) (Lotman, 1988, 2000; Werstch, 1998). Univocal discourse could be imagined with a conduit metaphor, with knowledge being sent in one direction. Even if verbal exchanges take place, the intention of univocal discourse is transmission of information. In contrast, dialogic discourse involves dialogue between at least two voices where the communication is used as a thinking device and new meaning is generated.

Along with functions of communication, structures associated specifically with classroom discourse are relevant to this research. For example, the basic structures of classroom discourse include the following: moves, exchanges, sequences, and episodes (Lemke, 1990; Mehan, 1985; Wells, 1999) (see Fig. 1). The move, exemplified by a question or an answer from one speaker, is recognized as the "smallest building block" (Wells, 1999, p. 236). An exchange is

[1] In addition to Pólya's use of the phrase "mathematics in the making," Speiser and Walter (2000) used the term as an adaptation of Latour's (1987) idea of "science in the making" (i.e., as contrasted with "ready-made science").

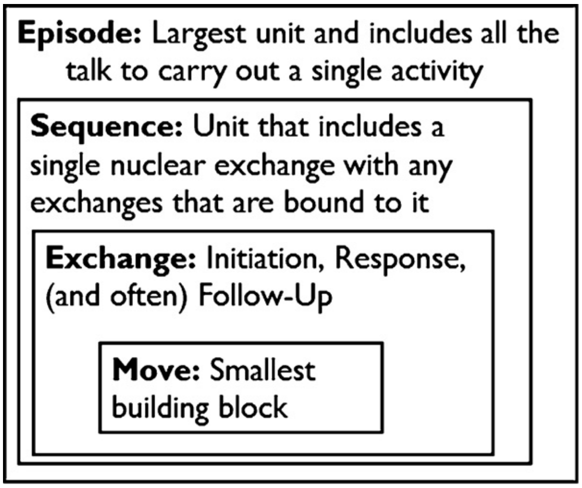

Fig. 1. Structures of classroom discourse.

made up of two or three moves and occurs between speakers (typically including initiation, response, and evaluation or follow-up moves: IRE or IRF). A sequence contains a single nuclear exchange (i.e., one that can stand on its own, independently contributing to the discourse) and any exchanges that are bound to it (i.e., bound moves depend on the nuclear exchange in some way). The episode is the level above sequence and represents "all the talk that occurs in the performance of an activity" (p. 237).

In addition to functions and structures, forms of talk and verbal assessment were identified from the literature and from the authors' previous research (Truxaw, 2004; Truxaw & DeFranco, 2004, 2005). The specific types of talk used in this analysis include the following: *monologic talk* (i.e., involves one speaker — usually the teacher — with no expectation of verbal response), *leading talk* (i.e., occurs when the teacher controls the verbal exchanges, leading students toward the teacher's point of view), *exploratory talk* (i.e., speaking without answers fully intact, analogous to preliminary drafts in writing) (Cazden, 2001), *accountable talk* (i.e., talk that requires accountability to accurate and appropriate knowledge, to rigorous standards of reasoning, and to the learning community) (Michaels, O'Connor, Hall, & Resnick, 2002). Two types of verbal assessment were also identified: inert assessment[2] (IA) and generative assessment (GA). IA is verbal assessment that guides instruction by maintaining the current flow and function of the discourse. GA, in contrast, is verbal assessment that mediates discourse to promote students' active monitoring and regulation of thinking. GA is associated with modeling and promoting metacognition (Flavell, 1979). Although it is not clear-cut, monologic talk, leading talk, and IA tend to be associated with univocal discourse and, in contrast, exploratory talk, accountable talk, and GA tend to be associated with dialogic discourse.

Both *talk moves* and *verbal assessment moves* represent verbal exchanges; however, they are distinguished in this research in order to more clearly delineate when teacher's feedback or interventions are taking place—making relationships between teacher's verbal assessment and students' talk moves more transparent. In this research, IA and GA moves typically involve *teacher's* verbal moves — either initiation or follow-up — that influence the flow and function of the talk. It is important to note that "assessment" is not synonymous with "evaluation"; rather, NCTM's (2000) description is informative: "Assessment should not merely be done *to* students; rather, it should also be done *for* students, to guide and enhance their learning" (p. 22). As referred to here, assessment could be evaluative, but could also include other types of feedback or intervention. Key differences between the two types of verbal assessment are how they affect the flow and function of the discourse; how they relate to student thinking; and their association with univocal or dialogic discourse. IA tends to maintain the status quo, keeping the flow and function of the discourse relatively constant (e.g., "Very good," or, "Okay")—typically, toward the teacher's point of view; whereas, GA often presses the discourse in new directions (e.g., "Why?" or "Could you explain your reason?")—potentially incorporating

[2] The authors based the term "inert assessment" on Alfred North Whitehead's description of "inert ideas"—that is, ones "that are merely received into the mind without being utilised, or tested, or thrown into fresh combinations" (Whitehead, 1964, p. 13).

students' views. IA is associated with conveying meaning (univocal); GA promotes elaboration and reflection, with potential for developing new and/or deeper understanding (dialogic).

The authors' previously reported research (Truxaw, 2004; Truxaw & DeFranco, 2004, 2005) demonstrated that graphic maps of discourse (called sequence maps) could be developed to represent the flow of the talk and verbal assessment moves and the overall function of the discourse (i.e., tending toward univocal or dialogic). Additionally, the sequence maps, when analyzed in conjunction with transcripts and other evidentiary data (e.g., interview transcripts and field notes), could be used to develop associated models of teaching. Thus far, three models of teaching have been reported: a deductive model (associated with univocal discourse), an inductive model (associated with dialogic discourse), and a mixed model (a hybrid of the other two). This paper describes how similar strategies and constructs were used as a basis for investigating the talk, verbal assessment, discursive functions, and associated teaching of a lesson taught by mathematics education expert, George Pólya.

3. Methods and procedures

The data derived from the MAA video, "Let Us Teach Guessing" (Pólya, 1966) (with permission from the MAA). The "participant" in this investigation was George Pólya, noted mathematician, problem solving expert, and mathematics educator. The lesson, filmed in 1965, was taught to university students in California. In the introduction to the video, Pólya explained his intentions for the lesson—that is, he wished to demonstrate his "attitude to teaching." In essence, "teaching is giving opportunity to the students to discover things by themselves." Additionally, he noted themes related to the lesson: "Finished mathematics consists of proofs, but mathematics in the making consists of guesses."

3.1. Coding and analysis

The dialogue from the video was transcribed, coded, and analyzed using strategies that had been developed in previously reported research (Truxaw & DeFranco, 2004, 2005). In particular, the transcripts were formatted into tables and numbered based on "utterances" (i.e., speaker's turns—from here-on-out to be called "lines") (Bakhtin, Holquist, & Emerson, 1986). *Moves* within each line of text were coded using strategies based on the work of Wells (1999) and Nassaji and Wells (2000) and adapted by the authors (see Appendix A). Table 1 illustrates lines 4–6 (of the 208 "lines"—i.e., utterances) of the lesson and represents a portion of sequence 4 (of the 19 sequences) in this lesson. Although all of the coded categories in Table 1 were used in analyzing the data, only those aspects necessary for the results of this study will be described. Briefly, the coding in Table 1 indicates that Pólya and his students were engaged in discourse following a triadic exchange structure (Initiation, Response, Follow-up — IRF — as seen in the "Mv" column). Lines 4 and 5 were coded as exploratory talk (see "Comment" column) because they were related to guessing (i.e., speaking without answers fully intact). The coding of line 6 indicated two verbal assessment moves. When Pólya repeated the student's response, saying, "25," it was coded as inert assessment (IA) since it was maintaining the status quo of the discourse. When Pólya followed up with, "How did you get it?" the move was coded as generative assessment (GA) because it pressed the student to explain her response. Similar schemes were used to code the other lines of text from this lesson.

Table 1
Example of coded text from Pólya's lesson

Ln	Seq	Who	Text	K1 K2	Exch	Mv	Pros	Func	Comment
4	4	P	Who is ready to say . . . *[inaudible]*? Don't be bashful. Go ahead. Yes. Say something.	K1	Nuc	I	D	Request info	Init Expl Tlk
5	4	Gl	Um, 25.	K2	Dep	R	G	Inform	Expl Tlk
6	4	P	25. How did you get it? *[writes it on the blackboard]*	K1	Dep	F	A D	Revoice Accept Req Just	IA GA

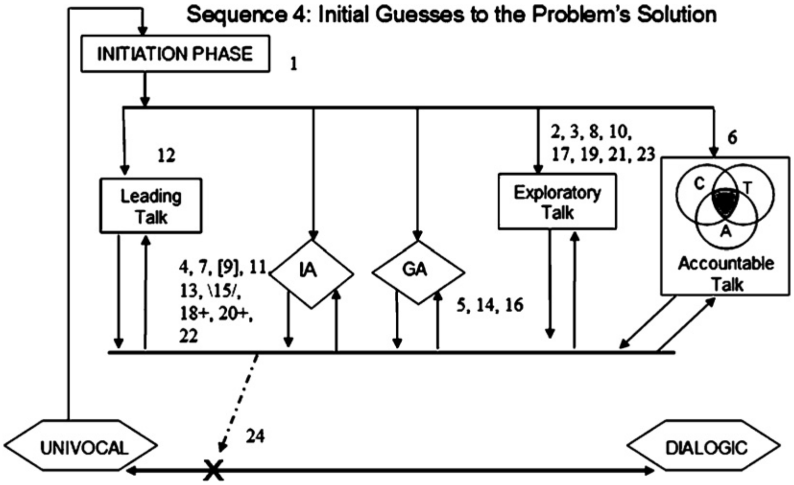

Fig. 2. Example of *sequence map* for sequence 4 in Pólya's lesson. The numbers represent consecutive verbal moves.

3.2. Mapping

After initial coding, the text was parsed into 19 sequences—that is, topically linked sets of verbal exchanges, as described in Section 2. Individual sequence maps (i.e., diagrams representing the flow of talk and verbal assessment moves within a sequence) were developed by applying the coded discourse moves to a graphic template of classroom discourse that had been developed by the authors in previous research (Truxaw & DeFranco, 2004) (see Appendix B for the template; see Fig. 2 for a sequence map developed for sequence 4 in this lesson). To translate the text to the map, each coded move within a sequence was renumbered beginning with the number 1 and ending with the last move in a particular sequence—for example, in Fig. 2, the number 1 represents the first move of sequence 4 within Pólya's lesson. Each number on the sequence map represents a verbal move—that is, either a type of talk or a form of verbal assessment. Consecutively following the numbered moves provides a map of the flow of the discourse. In addition to the indicators of the talk and verbal assessment moves, the diagram includes a dashed and dotted line that points to a marker X (Fig. 2, 24, X) placed along a line representing a continuum of discourse ranging from univocal to dialogic. It should be noted that the placement along the continuum is not definitive—it acts as a marker indicating overall *tendencies* of the discourse within a sequence toward univocal or dialogic. To determine univocal and dialogic tendencies, the coded transcripts were examined for indicators (informed by the literature and previous research) of the overall purpose of the discourse within a sequence.[3] The sequence represented in Fig. 2 was mapped as tending toward univocal since its *overall purpose* was for the participants to *convey information* (Lotman, 2000) about guesses being made. However, since there were some indicators of dialogic discourse (e.g., GA—moves 5, 14, and 16; exploratory talk—moves 2, 3, 8, 10, 17, 19, 21, and 23; and accountable talk—move 6), a slight shift along the continuum toward dialogic was shown.

3.3. Developing a model of teaching

To gain a more fine-grained view of the relationships between the types of talk and forms of verbal assessment and the functions of the discourse, the data were deconstructed in the following manner. First, the text from the lesson was divided into *sub-units*, according to natural, thematic breaks in the dialogue (e.g., explaining the problem, initial guesses, etc.). For each sub-unit, the following were analyzed: associated text from the transcripts, a map of the talk and verbal assessment moves, and a synopsis of the discourse. When sub-units corresponded with identified sequences, existing sequence maps were used; if they did not, then additional sub-unit maps were developed. The synopses of the discourse in the sub-units were supplemented with evidentiary data—in this case, from Pólya's published works (e.g., Pólya

[3] A list of indicators of univocal and dialogic discourse may be obtained from the authors upon request.

described the five planes problem in *Mathematics and Plausible Reasoning,* 1954a, and described suggested phases of teaching in *Mathematical discovery: On Understanding, Learning, and Teaching Problem Solving*, 1962/1981). The sub-unit analysis provided support for how the content, the flow, and the teacher's (i.e., Pólya's) intentions might influence the outcomes of the discourse. This deconstruction process achieved a fine-grained view; it also revealed that the sub-units needed to be considered within larger contexts (i.e., at the level of the *episode*) in order to develop a model of the teaching. The sub-units were then *reconstructed* within the context of the episode—a process that re-examined the sub-units and mapped these data onto components of a model of teaching aligned with the discourse found within the lesson. The components of the model will be described within Section 4.

3.4. Trustworthiness of coding and analysis

The strategies used for coding and analysis were developed and tested during a larger study involving multiple participants. Data collected from each participant were analyzed both individually and using constant comparison methods (Strauss & Corbin, 1990) so that each set of data would provide additional evidence to inspect, test, and refine the strategies and models being developed. Initial coding strategies were based on the established work of Wells (1999) and Nassaji and Wells (2000). Adaptations to the coding (e.g., attention to specific talk and assessment moves) were validated through the literature and through consultation with professional colleagues with expertise in both mathematics education and discourse analysis. Professional colleagues also served as peer debriefers (Lincoln & Guba, 1985) to provide trustworthiness and reliability of the coding and analysis of the data. For example, peer debriefers coded passages independently and brought them back for comparison and discussion with the researcher. If there were inconsistencies in coding, they were discussed until consensus was achieved. Additionally, peer debriefers worked together with the researchers to develop the components of the teaching models.

4. Results and discussion

Not surprisingly, initial viewing of the lesson showed expert teaching derived from deep understanding of both content and pedagogy. Further, the analysis of the discourse uncovered details that may help teachers move toward more reform-oriented practices. For example, the coding of the transcripts showed the use of triadic exchanges (IRF) to facilitate the discourse within the lesson. Although triadic exchange structure has been associated with "illusory understanding" (Lemke, 1990), Pólya demonstrated its use in conjunction with a rich mathematical problem and with discourse aimed at building (rather than simply conveying) students' understanding of mathematics. This is not a recommendation for use of IRF structures in classrooms; rather, it is an argument that the use of these structures does not preclude effective teaching and learning. An important question is: *how* did Pólya do this?

The analysis of this lesson yielded 19 sequence maps. The sequence maps allowed the researchers to count forms of talk and verbal assessment, but, more importantly, to indicate when and how each was used in the dialogue. The sequence maps showed the flow of the moves within each sequence as well as relationships of sequences to each other. For example, sequence maps 1–3 showed monologic talk that was univocal in nature. In contrast, the map for sequence 4 included leading, exploratory, and accountable talk and both IA and GA that, although univocal overall, had slight tendencies toward dialogic function (see Fig. 2). As the lesson progressed, the discourse shifted back and forth between exploratory stances (e.g., guessing) and talk that tended more toward conveying meaning. Increasing instances of accountable talk were documented as strategies were shared and applied. As will be described, GA was infused at critical junctures—typically when students had gathered sufficient evidence to question previous conjectures. While the flow and function of the sequences varied, the overall outcome of the lesson demonstrated teaching that did not "merely impart information" (Pólya, 1962/1981, p. 100), but, rather, encouraged building of "know-how" and new meaning.

4.1. Inductive teaching

Three models of teaching based on analysis of middle grades mathematics teachers were previously reported: deductive, inductive, and a mixed models (Truxaw & DeFranco, 2005). The model of teaching developed from Pólya's lesson most closely resembled the previously developed inductive model in that both moved from specific cases, through recursive cycles, toward more general hypotheses and rules. In order to enhance the readability and understanding of the

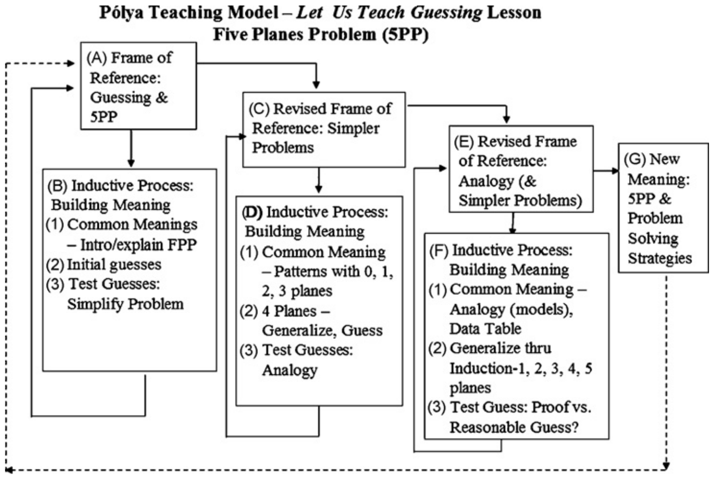

Fig. 3. Inductive model of Pólya's lesson.

model of teaching developed from Pólya's lesson (see Fig. 3), the components of the model (i.e., frame of reference, inductive processes, revised frame of reference, etc.) will be illustrated using selected classroom text, maps, and explanations that derived from multi-level analysis.

The first three sequences of the lesson established a *frame of reference* (see Fig. 3A) (i.e., the five planes problem [5PP] and the theme of guessing) and communicated *shared meaning* related to the problem and procedures (see Fig. 3 B-1). These will be illustrated with representative examples. In sequence 1 (see Fig. 4 for sequence map), Pólya used monologic talk to introduce the theme of the lesson—that is, guessing. A brief excerpt follows.

Pólya: I wish to teach you about guessing . . . Mathematics when it is finished, complete, all done, then it consists of proofs. But, when it is discovered, it always starts with a guess . . .

In sequence 2, Pólya again used monologic talk, this time to set the tone and share "rules" related to guessing.

Pólya: So we play together a guessing game. This, as any other game, has rules. The rules are very simple. There are just two rules: One, for those people who will know already my question, I hope there are very few, but if you know already my question . . . If you know already my problem, don't answer my question. That would be unfair, if you know already the answer . . . it wouldn't be guessing, and you would spoil the fun of all of us . . .

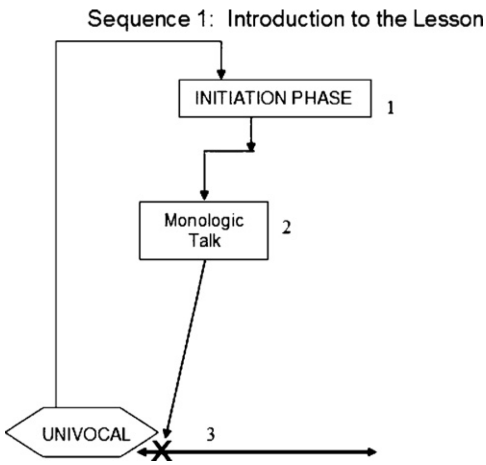

Fig. 4. Map for sequence 1 in Pólya's lesson.

Table 2
Coded excerpt of dialogue from sequence 4

Map#	Speaker	Text	Talk/assess
1	*Pólya:*	Who is ready to say *(inaudible)?*	Init
2		Don't be bashful. Go ahead. Yes. Say something.	ExplTlk
3	*S1:*	Um, 25.	ExplTlk
4	*Pólya:*	25.	IA
		How did you get it? *[Writes it on the blackboard.]*	GA
5	*S1:*	I looked at 5 times five.	AcctTlk
6	*Pólya:*	Five times five. That's an idea. There is an idea. Good. Anybody ready for another guess? Yes, please?	IA
7	*S2:*	32	ExplTlk
8	*Pólya:*	Thirty-two. Oh, oh. There's something behind your... Oh, 32. Interesting...	IA

> However, if you don't know the answer to my question, then don't hold back. See, don't be bashful, but guess. Of course, you guess may be wrong, but that's one of the odd points in the art of guessing: even a wrong guess is helpful . . .

In sequence 3, Pólya described the five planes problem [5PP]—that is, *into how many parts is space divided by 5 planes?* The maps for sequences 2 and 3 were essentially identical to that of sequence 1, showing monologic talk, no verbal assessment, and a tendency toward univocal discourse. It should be noted that throughout the lesson there were non-verbal aspects of Pólya's teaching that were not represented in the coding or the sequence maps, but that were noted in the sub-unit analysis. For example, as Pólya explained the problem, he employed the use of visual aids (the blackboard, the top of a desk, and a yard stick) and expressive language (e.g., describing cutting a block of cheese as an analogy to space cut by dividing planes). While the focus of the analysis was on the verbal discourse, non-verbal aspects of the teaching were noted.

In sequence 4, Pólya facilitated exploratory talk using both IA and GA. Table 2 provides a sense of how the dialogue connects to the sequence map. The numbered moves in the far left-hand column correspond to the numbered moves on the sequence map (see Fig. 2). These represent the talk and verbal assessment moves in the far right-hand column—in this case, exploratory talk, IA, and GA. The sequence map shows the flow of the verbal moves for the entire sequence and its overall tendency toward conveying meaning (i.e., univocal). This sequence corresponded with the *initial guesses* component of the model (see Fig. 3-B-2).

In sequence 5, Pólya introduced the strategy, "solve a simpler problem"[4] in order to begin to *test the guesses* (see Fig. 3-B-3). The original *frame of reference* (see Fig. 3-A) was revisited to facilitate moving toward a *revised the frame of reference* (see Fig. 3-C) that focused on simpler problems related to the original 5PP. A representative excerpt follows:

Pólya: So, if I ask you – five planes, then you should have asked yourself, why does he ask just five? Why not four? Why not six? So what would you ask? Yes?
S2: I guess, you mean, uh . . . How many planes do, say, three planes – spaces that three planes make?
Pólya: Good. Oh, what do you say?
S3: The simplest model would be two planes, I would think.
Pólya: Is that the simplest? (laughs)
Class: One.
Pólya: One! One plane. Oh, yes. So much trouble to find the simplest. One.

Sequence 5 included limited use of all four forms of talk and both IA and GA, with an overall tendency toward univocal discourse, but with a slight shift toward dialogic.[5]

[4] For a thorough description of the problem-solving heuristics in this lesson, see Leinhardt & Schwarz, 1997.
[5] *Note*: A complete set of sequence maps is available from the authors on request.

In sequences 6–10, Pólya facilitated investigation of related simpler problems—that is, space cut by one, two, and three planes. This exploration and discussion, along with recording data on the blackboard, helped to establish *shared meaning* about these simpler cases (see Fig. 3-D-1). Specifically, in sequence 6, Pólya explained space divided by one plane:

Pólya: So, here is . . . Here is for you one plane. *[Draws a line on blackboard]*. Oh, you tell me that is just one line on the blackboard. Yes, it is true. But, I mean in the following way: This line on the blackboard is the intersection of the blackboard with the plane, you see. By this plane I am showing you a horizontal plane. The horizontal plane could be a surface of quiet water of the reflecting pond – there is nothing else in the world, just this surface, and over it air, under it water. So how many parts?
Class: Two.
Pólya: Two. Good.

In sequence 7, Pólya drew a second line on the blackboard that intersected the first line. He used this to represent a second plane, helping the students to develop a shared understanding that space divided by two planes would yield four parts.

Sequence 8 diverged slightly when a student asked about the special case of parallel planes. Pólya noted that it was a "very good question," using this as an opportunity to say that the original problem was intentionally "incompletely stated" since "problems in life. . . are often incompletely stated." He then clarified the problem saying that the planes should be taken at random rather than parallel or passing through the same point.

In sequence 9, Pólya first asked the students to recall space divided by two planes and then illustrated the idea of space divided by three planes. He did this by noting the intersecting lines on the blackboard representing two planes, and the blackboard itself to represent a third plane. A representative excerpt of dialogue follows.

Pólya: . . . Now, look here, two planes indicated by the lines and the blackboard. There are—some of the parts are in the room in front of the blackboard. How many such?
Class: Four.
Pólya: Good. Some other parts are outside the room, behind the blackboard. How many such?
Class: Four.
Pólya: So all together, how many?
Class: Eight.
Pólya: Good. Eight.

In sequence 9, the verbal exchanges were predominantly of triadic (IRF) structure and the overall discourse tended toward univocal (conveying meaning); even so, the analysis demonstrated that Pólya used the verbal exchanges to model thinking skills and problem solving strategies. In sequence 10, Pólya introduced the idea of extreme cases—in this case, zero dividing planes, yielding one space. This was added to the existing data that had been recorded on the blackboard, as seen in Table 3.

The verbal exchanges, examples, and data table helped to build shared understanding of the simpler cases (i.e., space divided by zero, one, two and three planes) and provided means for noting potential patterns. This served as a basis to again infuse *guessing* (see Fig. 3-D-2) in sequence 11.

Table 3
Table from blackboard at the end of sequence 10

Dividing planes	Space/plane
0	1
1	2
2	4
3	8
4	

Pólya: Now, let me come to the next case. We have four dividing planes. Try to guess it. Four dividing planes. How many parts?

The consensus of the group, based on the observed patterns shown in Table 3 (i.e., one plane would divide space into two parts; two planes would divide space into four parts; three planes would divide space into eight parts), was that four planes would divide space into 16 parts. Although it would be uncovered later on that 16 was *not* the correct answer, Pólya did not dismiss the group's guess; rather, he reinforced the notion of a reasonable guess based on plausible reasoning.

Pólya: So we got really the 16 in a reasonable way of guessing. We observed. We found the pattern. And we said, and so on. It will go on like this. We made a generalization. That's very important, you see.

Pólya followed up by asking students to explain what they had observed to come up with their guess of 16. Students' responses further reinforced the idea of basing guesses on evidence.

In sequences 12–15, Pólya facilitated *testing the guess* (see Fig. 3-D-3) sharing another strategy—that is, *analogy*. Pólya suggested that the students consider *lines divided by points* and *planes divided by lines* as analogies to *space divided by planes*. This again *revised the frame of reference* (see Fig. 3-E). Although solving the 5PP was still a goal, the strategy of analogous situations provided a new way to consider patterns and relationships that might tie back to the original problem.

Pólya: Now, I will tell you, it is not very easy to, to imagine four planes at random in space . . . we can understand a little by the analogy of the plane. Analogy is another important point in guessing.

Pólya and the students worked through the analogous problems, incorporating two- and three-dimensional models, along with verbal discourse. For example, Pólya referred to two intersecting lines on the blackboard as a two-dimensional representation of a plane divided by two lines. He noted that, "all four parts are infinite, you see, not bounded." He continued, adding a third line to the drawing, creating a triangle, "But, if you introduce a third line, just at random, any position, like that, then something happens. Then there is one division of space more interesting than the others. It is bounded from all sides. It is finite." To further the discussion, Pólya showed a large, prepared drawing[6] that illustrated space divided by four planes and a physical model of a tetrahedron representing the finite space enclosed by four planes. Considering a bounded space (first the triangle and then the tetrahedron) provided means for students to reconsider their original thinking about the problem. This represented a critical juncture in the discussion, a time where infusion of GA seemed to have palpable potential.

As the discussion continued, Pólya reinforced the connection to analogies as a problem solving strategy—in particular, data related both to lines divided by points and planes divided by lines were noted and recorded. This led to considering the tetrahedron as means for counting the parts of space divided by four planes—the finite space (1) and the infinite spaces designated by the edges (6), vertices (4) and the faces (4) of the tetrahedron. The responses were discussed and recorded on the blackboard, thus helping to build further *shared meaning* (see Fig. 3-F-1) among the participants about related data. As a result, the students confronted their misconceptions and generated new meaning about space divided by four planes—that is, that *15 parts would be formed, rather than the originally conjectured 16 parts*. During this sequence, Pólya used both IA and GA to encourage the students to look for patterns and to generalize beyond the individual cases. GA, in particular, was instrumental in moving the discourse toward dialogic (i.e., generating new meaning), as seen on the map for sequence 14 (see Fig. 5).

In sequence 15, Pólya and the students worked through and recorded data related to divisions of lines, planes, and space, as represented by Table 4.

Sequence 16 represented another critical juncture for GA. In this sequence, Pólya challenged the students to revisit the 5PP, encouraging plausible guesses based on previous guesses and data that had been built and revised during the first 15 sequences.

[6] An example of the drawing used to represent space divided by four planes can be found in Pólya, 1954a, p. 45.

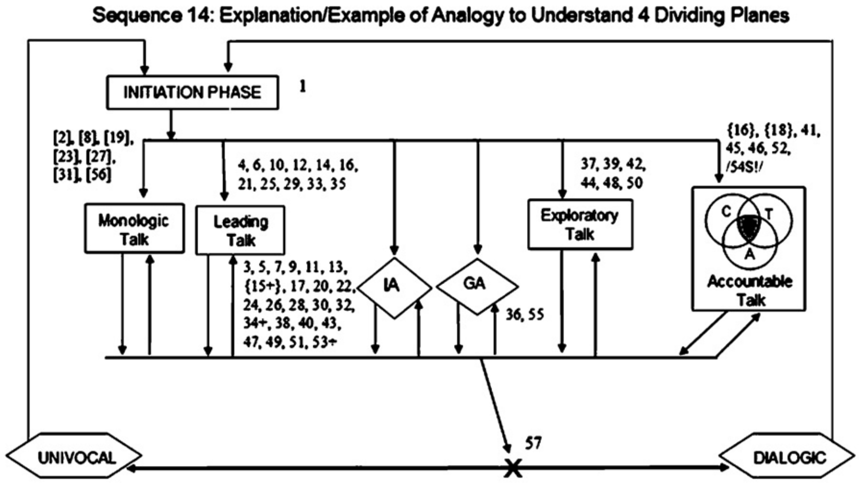

Fig. 5. Map for sequence 14 in Pólya's lesson.

Pólya: Now, we have little time, so I would be quite happy if you would solve my original question, which was: If you have five dividing planes, then space is cut into how many parts?

Students made several guesses including: 29, 28, 21, and 26. Then Pólya used GA to promote students' active monitoring and regulation of thinking (i.e., metacognition).

Pólya: . . .How do you come to 26? Could you explain your reason for 26?
S4: Yes.
Pólya: Yes?
S4: I was looking for a pattern in the numbers on the board. And I found that if you were working in the third column, that if you took the two preceding numbers in the other two columns . . . to get 8 you add the two 4's.
Pólya: *[Some overlap of speech as both P & S4 are talking]* In the third column? For instance, you take 4 *[points to 4 in third column on board]* . . . you take 4. Well, perhaps, or this column? *[points to another column]* What? Yes?
S4: Well, to get the 8 in the third column . . . I added. . .
Pólya: To get the 8. . .okay. . . that is the third row for me. Okay. *[Student had used the term column, when she meant row. P clarifies this.]*
S4: Yes, the third row.
Pólya: You add the two preceding?
S4: Yes, the 4s and the 4. Okay, to get that second 4.
Pólya: That is. . .above the 7?
S4: Okay. You add the number above it and the one over . . .
Pólya: Okay. Above the number 4.
S4: The two 2s.

Table 4
Table from blackboard at the end of sequence 15

Dividing	Space/plane	Plane/line	Line/point
0	1	1	1
1	2	2	2
2	4	4	3
3	8	7	4
4	15	11	5
5			

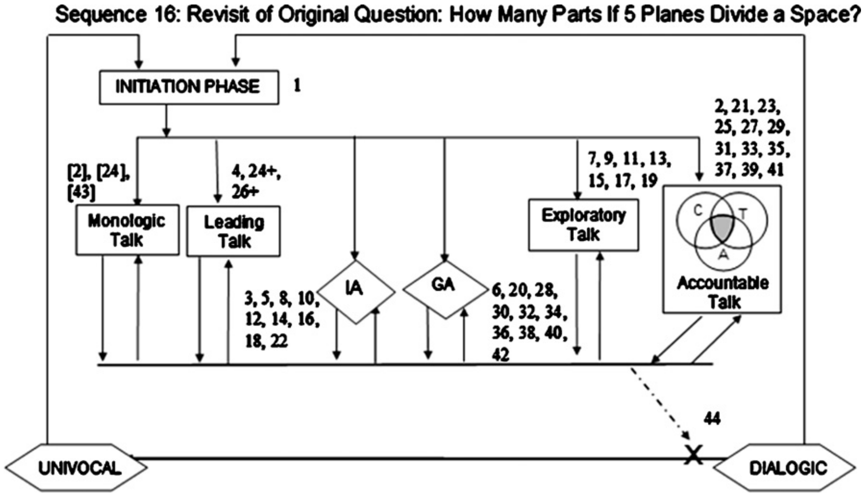

Fig. 6. Map for sequence 16 in Pólya's lesson.

Pólya: 2 and 2 are 4. 1 and 1 are 2. Seems to go. And 15. How did you get that?
S4: Okay. You add 8 and 7.
Pólya: Good. So, that is the pattern. It is correct. Now how did you get...? How did she get 26? Anybody else? Anybody follow? Yes?
S5: Add the 7 and 4 in the second and third column to get 11.
Pólya: *[Points to numbers indicated by student.]* 7 and 4, if the pattern continues, then it gets you 11.
S5: Then add the 15 and 11 in the first and second column.
Pólya: Then you get 26. Well, you see, this is a reasonable guess. A guess by induction. You see, it is based on observation, on observation of the pattern . . .

The students' explanations and ability to revise conjectures demonstrated that they had made connections among the strategies, patterns, and shared meanings that had been built throughout the lesson. Students hypothesized and defended their conjectures, using inductive *processes to build generalizations* (see Fig. 3-F-2). The group came to consensus that s*pace divided by five planes would form 26 parts*. This new meaning represented another example of dialogic discourse within this lesson (see Fig. 6 for the *sequence map* for sequence 16).

In sequences 17 and 18, Pólya facilitated a discussion about whether the answer (26 parts) had been *proven or, rather, was based on a reasonable guess* (see Fig. 3-F-3). The class agreed that while the guess of 26 parts was based on plausible reasoning, that it had not been proven. This, therefore, reminded them that it was important to *test guesses*. The strategies, themes, and *new understandings about the 5PP and the problem solving strategies* were summarized (see Fig. 3-G). Finally, in sequence 19, Pólya told a short story that reinforced the themes of guessing and proof.

The analysis revealed that the discourse in Pólya's lesson followed patterns representing an *inductive model of teaching* (see Fig. 3) that included recursive cycles, rather than linear series of steps. The lesson began with a frame of reference (i.e., the 5PP) and built meaning cyclically/recursively through inductive processes—that is, establishing shared meaning related to the problem and to problem solving strategies, conjecturing (i.e., guessing), investigating, and revising conjectures based on additional evidence. An answer based on plausible reasoning was developed through dialogic discourse, but the answer was not proved. The frame of reference of the lesson was the 5PP, but the outcomes of the lesson moved *beyond the solution* of the problem to include strategies and ways of thinking that could be applied to other mathematical problems—thus, mathematical meaning was developed.

4.2. Analysis of talk and assessment

Table 5 shows relationships of data from specific sequences, components of the inductive model, forms of verbal assessment and talk, and tendencies toward univocal and dialogic discourse. To determine the percentages shown on the

Table 5
Comparison of verbal assessment and talk moves and univocal versus dialogic tendencies

Sequences & Associated Components of the Model	Verbal Assessment		Talk				Univocal Dialogic
	IA	GA	Mono	Lead	Expl	Acct	◄—+—► U D
Seqs 1-3 – Frame of Ref. (5PP) & Shared Meaning	None	None	100%	0%	0%	0%	◄x—+——► U D
Seq 4 – Guessing, 5 planes	75%	25%	0%	10%	80%	10%	◄—x—+——► U D
Seq 5 – Intro Simpler Problem Strategy to Test Guesses	80%	20%	17%	33%	33%	17%	◄x—+——► U D
Seqs 6 -10 – Revised Frame of Ref: Shared Meaning (Simpler Problems, Patterns)	100%	0%	31 %	50%	11%	8%	◄x—+——► U D
Seq 11 – Guessing, 4 Planes	40%	60%	0%	0%	25%	75%	◄——x——► U D
Seq 12-15 –Test Guess (Analogy); Revised F of R: Shared Meaning (Patterns)	78%	22%	25%	41%	13%	21%	◄——x——► U D
Seq 16 – Revisit 5PP - Generalize from Analogies	47%	53%	12%	12%	28%	48%	◄——+—x► U D
Seq 17 – Test Guess	50%	50%	0%	33%	33%	33%	◄—x+——► U D
Seq 18 – Proof vs. Reasonable Guess	87%	13%	33%	33%	0%	33%	◄—x—+——► U D
Seq 19 – Closure (Story)	None	None	100%	0%	0%	0%	◄x—+——► U D
Overall	76%	24%	23%	33%	20 %	24%	

table, cumulative frequencies of each category of verbal assessment and talk were tabulated from the moves identified in the sequence maps. For each component, the total frequency of a *particular* verbal assessment move (e.g., IA) was divided by the total frequency of all verbal assessment moves within the associated sequence or sequences; similarly, the total frequency of a *particular* talk move (e.g., leading talk) was divided by the total frequency of all talk moves within the associated sequence or sequences. It is important to note that the percentages were based on frequency of moves, not length of time. Therefore, the percentages served as indicators, but should not be construed as being statistically significant. However, when combined with the other indicators, percentages of verbal assessment and talk moves help to reveal trends and patterns in the discourse. For example, sequences 1–3, associated with establishing the frame of reference, included no verbal assessment moves, exclusively monologic talk, and tended toward univocal. Sequence 4, associated with initial guessing, indicated the following: IA, 75%; GA, 25%; monologic talk, 0%; leading talk, 10%; exploratory talk, 80% exploratory; and accountable talk, 10%. The overall tendency of sequence 4 was toward univocal, but with a slight shift toward dialogic. In comparison, sequence 16, associated with revisiting the 5PP and making generalizing from analogies, indicated the following: IA, 47%; GA, 53%; monologic talk, 12%; leading talk, 12%; exploratory talk, 28%; and accountable talk, 48%. The overall tendency of sequence 16 was toward dialogic. In examining the patterns shown in Table 5, one can see that the lesson and the associated components of the inductive model began relatively univocally (sequences 1–10), shifted toward dialogic (sequences 11–16), and then shifted back toward univocal (sequences 17–19).

4.3. Instructional implications

Pólya has long been considered an expert in problem solving and mathematics teaching; the lesson described in this paper has been held up as a representation of his exemplary practice. The careful analysis of this lesson helps to legitimize what has been said anecdotally, providing evidence that Pólya used an inductive approach to teaching that was accompanied by strategic orchestration of verbal discourse. The resulting model can contribute to instructional practice by providing suggestions for how to teach in a Pólya-like way.

4.3.1. The model as a framework

First, the inductive model of teaching provides a framework. As used in the case of Pólya's lesson, a challenging problem was investigated through inductive teaching cycles. Basic components included: establishing a frame of reference; inductive processes of building shared meaning, guessing, and testing guesses; and then continuing similar cycles that incorporated revisions to the frame of reference and other processes. The model has the potential to scaffold Pólya-like instruction; knowledge of associated productive discursive strategies further enhances its usefulness.

4.3.2. Discursive strategies used in conjunction with the model

Discursive strategies can be examined and represented in multiple ways. For instance, the percentages of verbal assessment and talk moves shown in Table 5 are informative; however, *when* and *how* they were used further explicate Pólya's teaching practices. A case in point is the use of IA—often associated with transmission-style teaching and univocal discourse. While IA was the predominant form of verbal assessment used throughout the lesson, *IA was rarely used in an evaluative way*; rather, it was used as follow-up to keep the discourse moving—often to encourage exploratory talk (e.g., "That's an idea. . ."). GA was used much less frequently than IA, yet it appeared to be used strategically. In this lesson, typically, GA was infused after shared understanding of key ideas had been established. Further, the analysis indicated that GA was particularly productive in facilitating dialogic discourse when it was used in later repetitions of the cyclic process. A conjecture, based on the patterns of discourse, is that the GA was more productive when used after investigation had occurred and some shared meaning had been established. Revisiting Table 5, it is noteworthy that sequences that had stronger tendencies toward dialogic discourse also generally had relatively higher percentages of GA and accountable talk than those with tendencies toward univocal discourse (e.g., compare sequences 11 and 16 with sequences 5 through 10). In sum, dialogic discourse (and accompanying evidence of new meaning) seemed to be built through recursive cycles of establishing shared understanding (predominantly monologic talk, leading talk and IA), guessing (exploratory talk), infusing GA (pressing for metacognition) to test guesses (accountable talk) and moving beyond the guesses (toward "mathematics in the making").

4.3.3. Attention to metacognitive processes

Another feature of Pólya's teaching was attention to metacognitive processes (Flavell, 1979). For example, after students suggested a "guess" based on patterns of data, instead of simply accepting or rejecting it, he verbalized his thinking about their guess: "After you observed this pattern, you had some courage to say, 'and so on.' . . .so there is a step of generalization." In this case, his feedback provided a model for monitoring and regulation of thinking (i.e., metacognition). Pólya also used verbal assessment to set up an environment that supported these metacognitive processes. For example, speaker engagement was promoted when a student asked if he could add a guess and Pólya encouraged him with, "Oh, yes. Good. . . .Guesses are always accepted." When Pólya asked questions like, "What did you observe?" after students had made guesses, he facilitated examination not only of evidence, but also of decision-making. Questions like, "Could you explain how?" or "How did you get it?" encouraged explanations from students. Throughout the lesson, Pólya attended to metacognitive processes.

To teach in a Pólya-like way, this research suggests a strategic mix of univocal and dialogic discourse that, when used in conjunction with an *inductive model of teaching* and attention to metacognitive processes, may promote mathematical understanding in students. The teaching associated with the model illustrates how Pólya orchestrated classroom discourse that included some telling or *conveying of ideas* (univocal), but also discourse where the students and the teacher exchanged ideas in ways that resulted not simply in transmitting of ideas, but rather, in the generation of *new mathematical meaning* for some or all of the participants (dialogic).

4.4. Final remarks

Since its first publication in 1945, George Pólya's "How to Solve It" has been considered the classic text for problem solving. However, even with the insight and the clarity of the writing, translating the phases of problem solving into classroom practice has been a daunting challenge for teachers. The "Let Us Teach Guessing" video provides a bridge to link Pólya's writing and with classroom practice. Indeed, the point of this investigation was to not just solve problems, but to uncover Pólya-like practices for teaching mathematics.

The analysis of this lesson revealed two parallel planes of Pólya's expertise: first, Pólya demonstrated effective teaching of problem solving strategies through plausible reasoning and inductive processes; second, Pólya displayed an ability to orchestrate dialogue within inductive cycles that promoted dialogic discourse, and, as a result, mathematical meaning-making. Links between these two planes are the discourse and the inductive processes—this suggests how we might consider not only problem solving, but also more inclusive mathematical pedagogy. Using discourse within inductive cycles helped the students not only to develop problem solving strategies, but also to think mathematically in contexts beyond problem solving.

In the end, the inductive teaching facilitated by Pólya helped the students to build new meaning about the problem, about problem solving strategies, and about mathematical reasoning. While we would not suppose to suggest that an analysis of the discourse in Pólya's lesson can uncover the nature of his expertise or art, the belief is that some clues related to facilitating *mathematics in the making* have been revealed. In this way, we believe that this research contributes to a focus on learners through a focus on this exemplary teacher and scholar.

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Appendix A. Coding strategies for moves

Based on Wells (1999) and Nassaji and Wells (2000). Adaptations made by Truxaw (2004) underlined).

A	B	C	D	E	F	G	H	I	J
Line#	*Seq#*	Who	Text	K1/K2	*Exch*	*Move*	Pros	Func	Comment
214	13	T	Good job.	K1	Dep	I	A	Eval+	IA

The lettered columns represent the following:

A. Line # = Line number
B. Seq # = *Sequence* number
C. Who = Speaker – T = teacher; S = student (unidentified); Ss = students
D. Text = text of verbal discourse
E. K1/K2 = K1 = Primary Knower; K2 = Secondary Knower

F. *Exchange* = Type of *Exchange*
1. **Nuc** = Nuclear – contributes to substantive context
2. **Bound** = Bound – depends on nuclear *exchange* ("bound to it")
   a. **Dep** = Dependent – aspect of nuclear *exchange* developed through specification, exemplification, justification, etc.
   b. **Emb** = Embedded – deals with problems in the uptake of a *move* in the current hange, for example, a need for repetition
   c. **Prep** = Preparatory – further bound category, including acts such as bid-nomination in whole-class question-and-answer

G. *Move* = Type of *move*
1. **I** = Initiation
2. **R** = Response
3. **F** = Follow-up

H. Pros	= Prospectiveness – extent to which *move* determines later *moves*:
1. **D**	= Demand
2. **G**	= Give
3. **G+**	= Give with added tag making it more strongly prospective
4. **A**	= Acknowledge
I. Func	= Function
1. **Req act**	= Request action
2. **Req inform**	= Request information
3. **Req clarif**	= Request clarification
4. **Req expand**	= Request expansion/extension of previous contribution
5. **Req examp**	= Request example
6. **Req sug**	= Request suggestion
7. **Req opin**	= Request opinion
8. **Req explan**	= Request explanation (P = procedure; C = concept)
9. **Req justif**	= Request justification
10. **Req pos/neg**	= Request yes/no answer
11. **Req confirm**	= Request confirmation
12. **Req repeat**	= Request repetition
13. **Req restate**	= Request restatement of another's contribution
14. **Req explore**	= Request exploration
15. **Req Ag/Dis**	= Request Agree/Disagree
16. **Req observ**	= Request observation
17. **Req comp**	= Req comparison/contrast
18. **Req apply**	= Request application of concepts/procedures
19. **Act**	= Action (e.g, getting supplies, correcting work)
20. **Check**	= Check for understanding
21. **Chal**	= Challenge
22. **Bid**	= Request to speak
23. **Inf**	= Give information
24. **Sug**	= Give suggestion
25. **Opin**	= Give opinion
26. **Justif**	= Give justification/explanation
27. **Confirm**	= Give confirmation
28. **Qualify**	= Qualify previous contribution
29. **Clarify**	= Clarify own previous contribution
30. **Explan**	= Explain
31. **Extend**	= Extend/expand previous contribution
32. **Examp**	= Give relevant example
33. **Pos/neg**	= Give 'Yes' or 'no' answer
34. **Ag/Disag**	= Agree/disagree
35. **Observ**	= State observation
36. **Comp**	= State comparison/contrast
37. **Repeat**	= Repeat own previous contribution
39. **Apply**	= Apply concepts/procedures
40. **Nom**	= Nominate next speaker
41. **Acknowl**	= Acknowledge
42. **Accept**	= Accept previous contribution
43. **Reject**	= Reject previous contribution
44. **Eval**	= Evaluate previous contribution
	a. Eval+ = Positive
	b. Eval− = Negative
45. **Reform**	= Reformulate previous contribution
46. **Revise**	= Revise/Change own previous contribution
47. **Revoice**	= Speaker "revoices" another's contribution
48. **UpT**	= Uptake (when one conversant asks someone else about something the other person said previously (Nystrand, 1997).
49. **WT**	= Wait time (when teacher gives students time to think before answering)
50. **Sum**	= Summarize
51. **DK**	= Don't know (speaker expresses that he/she doesn't know)

J. Comments = Researcher's Comments

1. **Univ** = *Move* tends toward univocal
2. **Dialog** = *Move* tends toward dialogic
3. **Mono** = Monologic talk
4. **Lead** = Leading talk
5. **AT** = Accountable talk
   a. **AT-LC** = Accountable to learning community
   b. **AT-AAK** = Accountable to accurate and appropriate knowledge
      - **AT-AAK-F** = Fact
      - **AT-AAK-P** = Procedure
      - **AT-AAK-C** = Concept
      - **AT-AAK-V** = Vocabulary
   c. **AT-RT** = Accountable to rigorous thinking
      - **AT-RT-P** = Procedure
      - **AT-AAK-C** = Concept
6. **ET** = Exploratory talk
   a. **ET-P** = Procedure
   b. **ET-C** = Concept
7. **MT** = Metacognitive talk (associated with other forms of talk or assessment)
8. **AQ** = Authentic question (not prespecified information)
9. **QAQ** = Quasi-authentic questions
10. **TQ** = Test question (prespecified, known information)
11. **IA** = Inert Assessment
   a. **IA-P** = Procedure
   b. **IA-C** = Concept
   c. **IA-M** = Metacognition
12. **GA** = Generative assessment
   a. **GA-P** = Procedure
   b. **GA-C** = Concept
   c. **GA-M** = Metacognition
13. **FR** = Fact response
14. **PE** = Procedure explanation
15. **CE** = Concept explanation

Appendix B

Model of the Flow of Classroom Discourse (Truxaw, 2004; Truxaw & DeFranco, 2004).

B.1. Description of the model

The graphic representation of the model of discourse shows *possible* components and pathways, not necessarily what will occur in every mathematics lesson. The model serves as a template for creating sequence maps. The basic components of the model include the four forms of talk and the two forms of assessment. Within the model, GA stands for generative assessment, IA stands for inert assessment, and the A, T, and C within the Venn diagram represent the facets of accountable talk (i.e., **A** = **a**ccurate and appropriate knowledge; **T** = rigorous **t**hinking; and **C** = **c**ommunity). The lines indicate the flow of discourse. The dashed and dotted lines (that extend from the solid line connecting the forms of talk and assessment) indicate tendencies toward univocal or dialogic discourse. As the discourse progresses, its overall function typically tends more toward either univocal or dialogic. For example, if the discourse were predominantly univocal, it would tend toward the left side of the double-arrowed line. The placement along the double-arrowed represents a *tendency toward* one or the other, not an absolute position. The placement of the discourse more toward univocal or more toward dialogic is based on indicators that appeared within the coded transcripts from which the discourse had been mapped (Fig. B1).

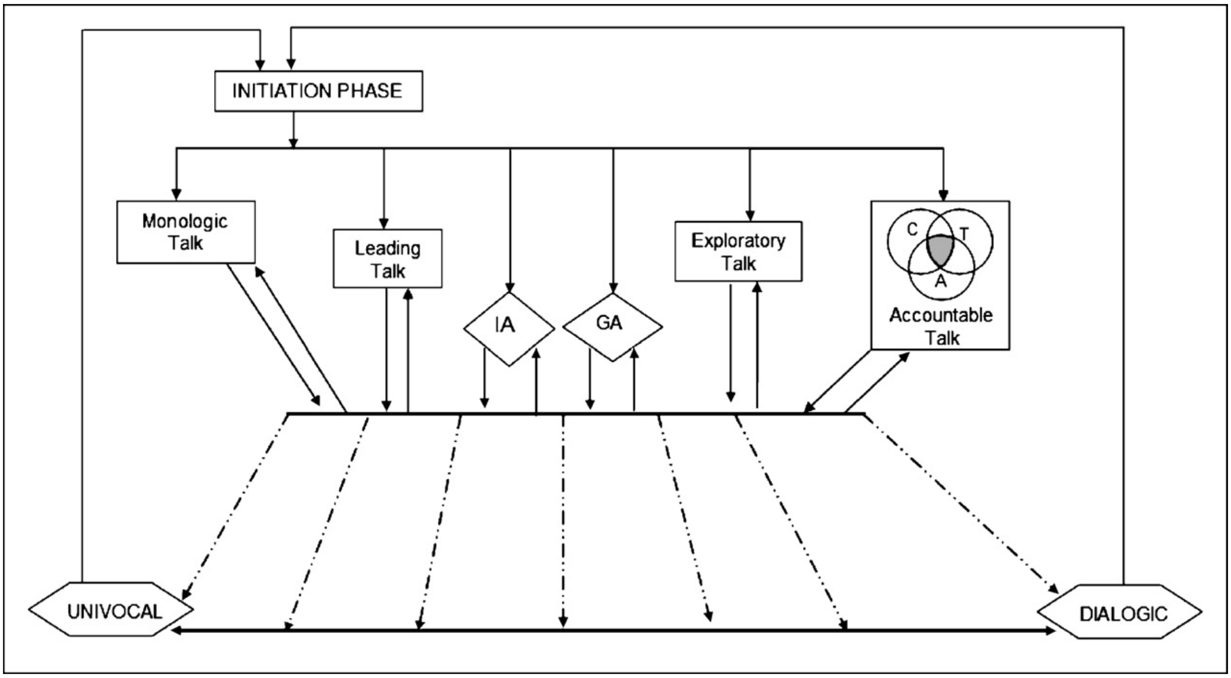

Fig. B1. Model of the flow of classroom discourse used as a template for developing sequence maps.

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