Lessons from
Mr. Larson

An Inductive Model of Teaching for Orchestrating Discourse

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The National Council of Teachers of Mathematics (NCTM) has consistently recognized communication as essential to reform-oriented mathematics teaching (NCTM 1991, 2000). "Through communication, ideas become objects of reflection, refinement, discussion, and amendment. The communication process also helps build meaning and permanence for ideas and makes them public" (NCTM 2000, p. 60). However, talking does not ensure that thinking and understanding follow. The quality and type of discourse affect its potential for promoting mathematical understanding (Kazemi and Stipek 2001).

In most mathematics classes, the teacher's questioning and feedback are used to convey information to students, leading toward the teacher's point of view. This type of discourse is referred to as *univocal discourse*. In contrast, dialogue that involves give-and-take communication in which students actively construct meaning is called *dialogic discourse* (Knuth and Peressini 2001). It has been found that the teacher's role is critical not only in how the discourse plays out in a mathematics class but also in outcomes of student learning (NCTM 1991, 2000). For example, recent evidence suggests that simply increasing the quantity of student talk may not improve mathematical understanding

since students may not have the resources to construct or verify mathematical ideas or conventions (Nathan and Knuth 2003). There are times when the teacher may need to step in (Rittenhouse 1998); indeed, there are those who argue convincingly that "telling" should not be eliminated from teachers' repertoires (Lobato, Clarke, and Ellis 2005).

In this article, we propose a strategic mix of univocal and dialogic discourse that, when used in conjunction with an *inductive model of teaching,* can promote mathematical understanding in students. The inductive model was developed from analysis of one purposefully selected teaching episode that took place in an eighth-grade algebra class taught by Mr. Larson (all names in this article are pseudonyms), a nationally certified teacher with thirty-five years of teaching experience. The description of the teaching associated with the model illustrates how Mr. Larson orchestrated classroom discourse that included some transmitting, or "telling," of ideas (univocal discourse) as well as discourse in which students and teacher exchanged ideas. This latter discourse resulted in the generation of new mathematical meaning for some or all of the participants (dialogic discourse).

USING FEEDBACK AND QUESTIONING IN CLASSROOM DISCOURSE

The discourse in Mr. Larson's mathematics class was influenced by his use of verbal feedback and questioning. For example, when Mr. Larson followed up an interesting result by asking, "I wonder if that's always true," he modeled metacognition—that is, the monitoring and regulation of thinking (Flavell 1979). To promote mathematical meaning making, Mr. Larson interspersed metacognitive suggestions at pivotal points throughout the episode. Other types of feedback and questioning set up an environment that could support these metacognitive processes. For example, "Nice job" and "Yes, that makes sense" reinforced speaker engagement. "What is an abundant number?" sought an explanation or definition. "Why?" encouraged further exploration or justification. The dialogue is annotated to illuminate goals.

DISCOURSE IN MR. LARSON'S MATHEMATICS CLASS

When asked about the teaching episode, Mr. Larson said that he had been motivated by an interesting problem he had seen in a mathematics competition:

Find the sum of the reciprocals of all the factors of 28 (i.e., $1/1 + 1/2 + 1/4 + 1/7 + 1/14 + 1/28 = 56/28 = 2$).

While working through the problem himself, Mr. Larson had noticed a relationship to perfect numbers. His own curiosity led him toward goals for
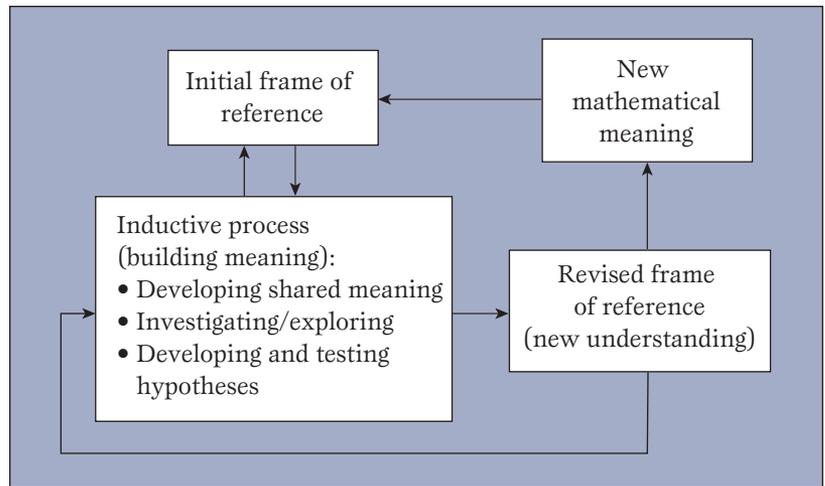

Fig. 1 Inductive model of teaching

his students—that is, getting his students to "think about numbers, think about relationships, be surprised by something, and then . . . test it out."

To guide discovery, Mr. Larson adopted an inductive model of teaching (see **fig. 1**). First, he introduced a slightly modified problem to his class:

What is the sum of the reciprocals of the prime or composite factors of 28?

This problem served as an initial frame of reference from which the rest of the episode was built.

Next, Mr. Larson facilitated a discussion to establish shared meaning about the frame of reference. The discussion was used to clarify the problem and ensure shared understanding of the vocabulary necessary for the problem—that is, *prime numbers* and *composite numbers*. He encouraged students to express their own understanding by using verbal feedback and questioning to promote an accurate understanding of the definitions. For example, when a student offered as a definition of prime numbers "Numbers that can be divided by only one and itself," Mr. Larson facilitated an exchange of ideas using examples and counterexamples to illustrate that the number 1 is neither prime nor composite. As a result, a student restated the definition to include this qualifier: "It has exactly two different factors." Ensuing dialogue helped the students agree on a more precise definition of prime numbers. Similar discussion was used to reach consensus on a definition of composite numbers.

Mr. Larson then asked the students to work in small groups to investigate the problem. During the small-group work, Mr. Larson listened and observed, identifying students who might need assistance and also those whom he might call on to contribute to a meaningful discourse during the large-group discussion. When the whole group reconvened, a student volunteered to share his solution.

David. First, I wrote out all the factors I knew of 28 [*lists 1, 2, 4, 7, 14, and 28*].

Mr. L. Yes. This makes complete sense, to start with a list of all the possible factors. [*This comment reassured the student (and the rest of the class) that the process he used was mathematically sound and confirmed the accuracy of his list.*]

David. Then I figured out the prime ones; 2 and 7 are the prime factors. And 4, 14, and 28 are the composite ones [*circles numbers as he says them*]. So I turned those into their reciprocals [*shows reciprocals: 1/2, 1/4, 1/7, 1/14, 1/28*]. And then for easier adding, I just flipped them all into 28ths [*shows equivalent fractions with denominators of 28: 14/28, 7/28, 4/28, 2/28, 1/28*].

Mr. L. Yeah. [*This response signified that Mr. L. was following the explanation and that the speaker should continue.*]

David [*adding the fractions on the white board*]. And then when I added them up I got 28/28, which is 1.

After the student completed his explanation, the class, by consensus, agreed that the sum of the reciprocals of the prime and composite factors of 28 equals 1.

Mr. L. That's sort of surprising that it would actually be 1. I wonder if that's always true. [*This comment modeled metacognition, that is, it provided a window into Mr. L.'s thinking about mathematics and encouraged the students to monitor their own thinking.*]

Mr. Larson then suggested that the students try a different number to test this idea.

Mr. L. I'm going to try another. I'm going to try 6. Somebody said something about 6. [*Mr. L. knew that 6 would work because it, like 28, was a perfect number. This helped set up a situation to serve as a springboard for further exploration and discussion.*]

Next, Mr. Larson guided classroom discussion to demonstrate that the sum of the reciprocals of the prime or composite factors of 6 also equals 1. At this point, Mr. Larson orchestrated the introduction of a hypothesis related to the problem.

Mr. L. Whoa! So what should we call this? Should we call this the Hankins Hypothesis or what? [*Hankins was David's last name.*] You want credit for it, David? [*Mr. L.'s "surprised" reaction marked the result as noteworthy and was, in a limited sense, modeling metacognition. Using the student's name reinforced his contribution.*]

David. Definitely.

On the board, Mr. Larson wrote the Hankins Hypothesis: The sum of reciprocals of the prime and composite factors of a number will always be 1.

[*Although Mr. L. knew that only perfect numbers would yield a result of 1, the "hypothesis" provided a vehicle for further investigation. In his interview, Mr. L. noted that he hoped that his students would discover that the "hypothesis" worked only for perfect numbers.*]

Next, the frame of reference was revised to consider not only the original problem but also new understandings that had been discussed—that is, the Hankins Hypothesis. The inductive process continued as Mr. Larson asked students to work in small groups to test the hypothesis.

Mr. L. We've seen two examples now where it works. I'm sort of surprised . . . I don't know why it would work, but it seems to work . . . [*Mr. L. modeled metacognition as he reflected on the mathematics, thus encouraging students to monitor their own thinking.*]

Mr. L. Would you guys check it out? Would you each take some other number and check it to see if, in fact, it does work? [*This suggestion underscored further exploration.*]

Mr. Larson circulated among students, listening and asking questions. When a student said, "It doesn't work for primes," Mr. Larson asked him to think about why that might be so. Mr. Larson called the class's attention back to the whole group and asked, "Okay, so what did you discover?" When students reported specific numbers that did not work, Mr. Larson, instead of abandoning the hypothesis, suggested that these cases might be exceptions to the Hankins Hypothesis. Exceptions to the hypothesis—for example, that the hypothesis does not work for primes or that the hypothesis does not work for perfect cubes—were documented on the board and named after the students who offered them. The verbal exchanges continued until a student said, "It didn't work for 36, which is an abundant number."

Mr. L. [*dramatically*]. Whoa! A what?

David. An abundant number.

Mr. L. An abundant number! What is an abundant number? [*This query sought an explanation of a term that was unfamiliar to many of the students and helped facilitate shared meaning. In addition, it provided an opportunity for peer teaching.*]

David. When the factors of the number add up to more than the number itself.

A discussion followed that clarified the definition of *abundant numbers* and, further, led to the introduction of and discussion about *deficient numbers* and *perfect numbers*. In summary, the Hankins Hypothesis provided a revised frame of reference that served as a basis for exploring connections between the original problem and other mathematical ideas—that is, abundant, deficient, and perfect numbers.

Students verbalized connections between numerical concepts, the hypothesis, and the original problem. For example, a student made a connection between his newly acquired knowledge of deficient numbers and the Hankins Hypothesis. In addition, students were able to identify the two numbers that worked for the original problem (i.e., 28 and 6) as perfect numbers.

Mr. L. Perfect number. Well, anybody know any perfect numbers? Daniel? [*This question encouraged connections with the original problem, the hypothesis, and perfect numbers.*]
Daniel. Six. [*Recall that the number 6 worked for the Hankins Hypothesis.*]
Mr. L. Six is a perfect number. Huh! . . . and . . . Arthur? [*This comment reinforced connections and validated the student's response.*]
Arthur. Twenty-eight. [*Recall that 28 was the number introduced in the original problem.*]
Mr. L. Twenty-eight is a perfect number. Mmm-mmm. [*Again, this response reinforced connections and validated the student's response.*]

In the end, the Hankins Hypothesis was modified to incorporate the revised understanding.

Mr. L. David, would you like to modify the Hankins Hypothesis? [*This invitation gave authority to the student and facilitated connections among the initial frame of reference and the revised frames of reference.*]
David. They have to be perfect numbers, not just any number.
Mr. L. Let's see here! . . . So the sum of the factors of prime and composite [*reads from board as he writes*] . . . sum of the reciprocals of prime and composite factors of a perfect number will be one. [*This statement summed up the connections made within the episode.*]

The episode concluded with Mr. Larson challenging the students to find the next perfect number and see if it fit the newly revised Hankins Hypothesis. The verbal interactions resulted in new mathematical meaning being voiced by the students. Before this lesson, most of the students in the class had been unfamiliar with abundant, deficient, and perfect numbers. They now had a sense of their prop-

erties. Perhaps more important, the students had developed new meaning about mathematical exploration and discovery. As noted in Mr. Larson's previously stated goals, they were able to "think about numbers, think about relationships, be surprised by something, and then . . . test it out."

AN INDUCTIVE MODEL OF TEACHING

The analysis of Mr. Larson's classroom dialogue revealed an inductive model of teaching that promoted dialogic discourse. Mr. Larson used verbal feedback and questioning to move students through recursive, inductive cycles rather than through a linear set of steps. In the first cycle, a rich problem was introduced (providing a frame of reference), shared meaning of necessary components of the problem (e.g., vocabulary) was established, the problem was explored, the problem was solved, and then the problem served as a springboard for developing and testing a hypothesis. The cycles continued recursively, building from the outcomes of the first cycle. The solution to the problem and the accompanying hypothesis served as a revised frame of reference, the hypothesis was investigated and tested, new concepts were introduced, shared meaning of the new concepts (e.g., abundant numbers) was established, connections were made to the original problem, the hypothesis was revised, and new mathematical meaning was generated. To summarize, the inductive model includes recursive cycles that use a frame of reference as a foundation to establish shared meaning, to investigate, to conjecture, and to build new meaning progressively. It is called inductive because it moves from a specific case, through conjectures, toward more general hypotheses, rules, and relationships.

The richness of the discourse in Mr. Larson's mathematics class was a result of using both univocal discourse (conveying ideas) and dialogic discourse (generating new meaning) to build meaning progressively, with an overall outcome that tended more toward dialogic. Mr. Larson orchestrated discourse that established necessary shared meaning but then strategically infused metacognitive feedback and questioning that pressed toward new understandings. Although it is likely that other models of teaching could promote mathematical understanding, the inductive model seems a

> **Goals for students included getting his students to "think about numbers, think about relationships, be surprised by something, and then . . . test it out"**

promising one to consider, especially when used in conjunction with thoughtfully orchestrated discourse.

IMPLICATIONS FOR CLASSROOM PRACTICE

Certain practices and dispositions associated with Mr. Larson's teaching have implications for classroom practice. A few examples follow.

- The problem resulted from Mr. Larson's curiosity and his practice of *doing* mathematics himself. A teacher's disposition toward mathematics may influence his or her classroom dialogue and, as a result, how the mathematics is taught and how it is learned. In effect, a teacher's disposition may influence students' learning outcomes.
- Mr. Larson's stated intentions were aligned with dialogic outcomes—that is, his goal was to facilitate "guided discovery." Although intentions are not a guarantee of success, what a teacher intends does matter. For example, Mr. Larson was more likely to achieve dialogic discourse through guided discovery than would a teacher with univocal intentions.
- Mr. Larson facilitated discourse that moved back and forth between whole-group and small-group talk. Meaningful discourse may benefit from orchestration of various talk formats.
- The discourse moved from relatively univocal (while building shared meaning) to relatively dialogic (as new meaning was generated). It may be productive to build shared meaning about the mathematics before pressing toward metacognitive processes. Once shared meaning is established, strategically interspersing metacognitive feedback and questions may help students monitor their own thinking actively—potentially promoting dialogic discourse and mathematical understanding.
- The solution to the problem was not an end in itself; it served as a springboard for moving beyond the problem, making connections to mathematical concepts and ways of thinking. As noted by NCTM (2000), "Interesting problems that 'go somewhere' mathematically can often be catalysts for rich conversations" (p. 60).

The cyclical-recursive nature of the discourse allowed the students to explore a problem, develop hypotheses, test them, revise them, and build mathematical understanding. Pirie and Kieren (1989) noted that mathematical understanding "is a recursive phenomenon" and occurs "when thinking moves between levels of sophistication" (p. 8). The recursive cycles of the inductive model may facilitate mathematical understanding.

NCTM has noted that communication is an essential component of mathematics teaching and learning. We believe that there are lessons to be learned from Mr. Larson that may provide hints for orchestrating discourse that supports mathematical understanding.

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